A Numerical Evaluation of the Sustainable Size of the Primary Deficit in Japan

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Abstract

We investigate how large a primary deficit-to-GDP ratio Japan's government can sustain. For this investigation, we construct an overlapping generations model in which multi-generational households live and the government maintains a constant ratio of the primary deficit to GDP. We numerically show that the primary deficit cannot be sustained unless the rate of economic growth is unrealistically high, which is more than five percent according to our settings. Our result implies that Japan's government needs to achieve a positive primary balance in the long run in order to avoid the divergence of the public debt-to-GDP ratio.

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Key Words: fiscal sustainability; public debt; primary deficit; economic growth.

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1 Introduction

Since the 1980s, the amount of public debt in Japan has ballooned. The financial liabilities of the general government in Japan increased from 52.1% of GDP in 1980 to 232.8% in 2011[1]. Other developed countries such as EU member countries and the U.S. also face the problem of public debt accumulation. Under these circumstances, fiscal sustainability has become one of the most important macroeconomic and policy issues, and many researchers have studied it. For example, Ihori et al. (2003) empirically investigate sustainability of Japanese fiscal stance by analyzing whether the intertemporal budget constraint of the government holds[2]. They conclude that, in Japan, fiscal consolidation is necessary to avoid the divergence of the ratio of public debt to GDP.

Meanwhile, theoretical literature exists on the dynamics of public debt and fiscal sustainability with a primary fiscal deficit. Previous studies in this body of literature provide implications about conditions preventing the ratio of public debt to GDP from diverging infinitely[3]. Chalk (2000) constructs a simple overlapping generations model and shows that the public debt-to-GDP ratio converges to some finite level with the primary deficit if and only if (i) the rate of economic growth is higher than the interest rate on public debt in steady states, (ii) the primary deficit is sufficiently small, and (iii) the initial amount of public debt is also sufficiently small compared with that of physical capital. Chalk (2000) then calibrates the model to match the U.S. economy from 1954 to 1980 and quantitatively derives an upper limit of a sustainable primary deficit. He concludes that the maximum size of a sustainable primary deficit is 5.2% of GDP even if the interest rate on public debt is below the rate of economic

[1] These data are obtained from Japan's SNA national account data.

[2] This approach to evaluating fiscal sustainability has been developed and used in many previous studies such as Hamilton and Flavin (1986) and Chalk and Hemming (2000).

[3] For example, Bräuninger (2005), Yakita (2008), Arai and Kunieda (2011), and Arai (2011) analyze the dynamics of the public debt-to-GDP ratio in endogenous growth settings.

growth in steady states.

This paper aims to answer the following questions based on Chalk's analysis: *Under what conditions can Japan's government run a primary deficit? How large a primary deficit-to-GDP ratio can Japan's government maintain?* It should be noted that Chalk's calibration is based on U.S. data during a term when economic growth rates were significantly higher than government bond interest rates and that the maximum deficit calculated in his analysis is highly sensitive to calibrated parameters. We need to implement a deliberate calibration matched with recent Japanese data in order to apply the analysis to the current issue of Japan's fiscal sustainability. We also extend the model more practically to examine the limit of the level of government expenditure by introducing actual revenue composition, effects of distortion by a tax system, and a realistic saving behavior of households.

In order to answer the questions, we construct an overlapping generations model similar to the one used in Chalk (2000). The model assumes a closed economy and a constant ratio of the government's primary fiscal balance to GDP[4]. We introduce exogenous differences between the interest rates on public debt and the rates of return on physical capital in order to replicate the actual gap between them[5]. Given the level of physical capital per GDP, this difference decreases the interest rate on public debt, and thus, affects the sustainable size of primary deficits[6].

[4]The assumption of a closed economy has been commonly adopted in the analysis of Japan's fiscal sustainability based on overlapping generations models such as Ihori et al. (2006), Braun et al. (2009), and İmrohoroğlu and Sudo (2011). In reality, Japan is an open economy, and increased government borrowing does not necessarily directly decrease real investment. The rationale for assuming a closed economy is the existence of the home bias, which hampers smooth international financial flows. Hoshi and Ito (2012) mention the strong home bias of institutional investors in Japan owing to high currency risk and capital adequacy requirements. Arslanalp and Tsuda (2012) observe increased demand from domestic financial institutions for their own government debt to be a global phenomenon, reflecting bank deleveraging under new financial regulations and rising home bias.

[5]For example, Ueda (2012) explains that, in Japan, returns on physical capital have been around 4.5–5.0 % since 2000, while the interest rates on public debt have been around 1.5 %.

[6]Sakuragawa and Hosono (2010) emphasize the effect of the difference between interest rates.

Key parameters are calibrated so as to match the data for the Japanese economy in 2005, and the growth rate of labor productivity, which is the engine of economic growth in the model, is exogenously given. Under the calibrated parameters, we show that not even a small primary deficit can be sustained in the long run, unless the economic growth rate continues to attain an unrealistically high level. In our benchmark case, the rate of economic growth must be more than 5% for a steady state to exist. In other words, if the economic growth rate is below 5%, the ratio of public debt to GDP diverges infinitely as long as the government maintains its primary deficit. We note that this condition is necessary but not sufficient to guarantee fiscal sustainability. As Chalk (2000) shows, the ratio of public debt to GDP does not diverge if and only if a steady state exists *and* the initial amount of public debt is sufficiently small. In fact, a growth rate higher than 5% may be necessary to avoid such divergence depending on the initial amount of public debt. Furthermore, we check the robustness of our results with respect to the alternative intertemporal elasticity of substitution, since previous empirical studies estimate various values.

Our results imply that it is necessary for the Japanese government to achieve a primary fiscal balance in order to avoid the divergence of the ratio of public debt to GDP. In other words, we consider that it is overly optimistic to think that Japan's fiscal sustainability can be guaranteed by a significant economic boost without eliminating the primary deficits.

A number of previous studies have a similar motivation to ours. In particular, this paper is related to Ihori et al. (2006) and İmrohoroğlu and Sudo (2011). Ihori et al. (2006) quantitatively find the tax rates and contribution rate needed to maintain current social security systems, such as public pensions and medical insurance, as well as to achieve an exogenously given target level of public debt-to-GDP ratio in the future. We do not levy any constraints on the level

In their paper, the difference is considered by introducing the financial intermediation cost.

of public debt and give the *set* of the size of the primary balance that prevents the public debt to GDP ratio from diverging infinitely. Furthermore, we also consider a labor–leisure choice and the endogenous retirement of households, which are not considered in Ihori et al. (2006).

İmrohoroğlu and Sudo (2011) show the necessary economic growth rate to prevent Japan's public debt-to-GDP ratio from diverging by using a standard neoclassical growth model. While their motivations are similar to ours, our approach is qualitatively different from theirs. They take future government expenditures, including that of interest payment on public debt, as exogenous variables. In such a setting, economic growth reduces the relative size of government expenditures and the fiscal deficit in relation to GDP. In this paper, we adopt an overlapping generations setting in which the interest rate on public debt is endogenously determined and the government's primary deficit relative to GDP is constant under different economic growth rates.

The remaining part of this paper is constructed as follows. In Section 2, we analyze a simple overlapping generations model in which households live for only two periods in order to understand the properties of the dynamics of public debt. The discussion here is based on that of Chalk (2000). In Section 3, we construct an overlapping generations model with multiple generations and show the results of calculating the maximum sustainable primary deficit to GDP ratio. Section 4 gives the results of our numerical calculations and discusses them. Section 5 discusses extensions of the model, and Section 6 concludes.

2 A Simple Overlapping Generations Model

Before our numerical evaluation on the sustainable size of primary deficit, we review the theoretical results shown in Chalk (2000). The purpose of this section is that we understand the properties of the dynamics of public debt in our model.

We consider an overlapping generations economy in which households live for two periods (young and old). There is no uncertainty and intra-generational heterogeneity. The size of the population of each generation grows at a rate of $1 + n$. When households are young, they supply their labor inelastically and consume and/or save their wage income. When old, they deplete and consume their savings. Households born at t have an identical utility function given by[7]:

$$\frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{d_{t+1}^{1-\sigma}}{1-\sigma}, \tag{1}$$

where $\beta \in (0, 1)$ is the subjective discount rate, σ is the inverse of the intertemporal elasticity of substitution, c_t and d_{t+1} are their consumption when young and old, respectively. Households face intertemporal budget constraints as follows,

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t, \tag{2}$$

where R_{t+1} is interest rate and w_t is wage rate, respectively. Each household maximizes the lifetime utility (1) subject to intertemporal budget constraint (2). By solving the maximization problem, we obtain the amount of saving, s_t, as follows,

$$s_t = \frac{1}{1 + \beta^{-1/\sigma} R_{t+1}^{1-1/\sigma}} w_t. \tag{3}$$

A representative firm produces final goods by using labor and physical capital in a perfectly competitive market. A production function is given by $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where Y_t is output, A is a scaling parameter, K_t is an aggregate amount of physical capital, L_t is an aggregate amount of labor supply, and $\alpha \in$

[7]Even if household's utility function is assumed to be a more general form, qualitative results do not change. See Chalk (2000).

$(0, 1)$ is an exogenous parameter, respectively. Then, the profit maximization conditions are:

$$r_t = A\alpha K_t^{\alpha-1} L_t^{1-\alpha}, \tag{4}$$

$$w_t = A(1-\alpha) K_t^{\alpha} L_t^{-\alpha}. \tag{5}$$

A government keeps the ratio of primary deficit, D_t, to GDP constant forever. That is, for all t,

$$\frac{D_t}{Y_t} = D. \tag{6}$$

Note that D is exogenously given. Government expenditure is supposed to be wasteful: public spending does not directly affect households' utility and productivity of final good production. A government follows the budget constraints in every period,

$$B_{t+1} = R_t B_t + D_t, \tag{7}$$

where B_t is an amount of public debt in the beginning of period t, and R_t is interest rate on public debt, respectively.

Because we consider a closed economy, all markets must clear in all periods: the labor market clearing condition is $L_t = N_t$ and the capital market clearing condition is $K_{t+1} = s_t N_t - B_{t+1}$. Furthermore, we suppose that a no-arbitrage condition holds in a competitive equilibrium: for all t, $R_t = 1 + r_t$.

Lastly, we define a competitive equilibrium and a balanced growth path steady state. Given the initial level of public debt, B_0, the initial level of physical capital, K_0, and the primary deficit to GDP ratio, D, a set of sequences of predetermined variables $\{K_t, B_t\}_{t=1}^{\infty}$ and of price system $\{R_t, w_t, r_t\}_{t=0}^{\infty}$ constitutes a *competitive equilibrium* if they satisfy the conditions as explained above for all t. Furthermore, given the primary deficit to GDP ratio, D, a set of sequences of predetermined variables $\{K_t, B_t\}_{t=1}^{\infty}$ and of price system $\{R_t, w_t, r_t\}_{t=0}^{\infty}$ constitutes

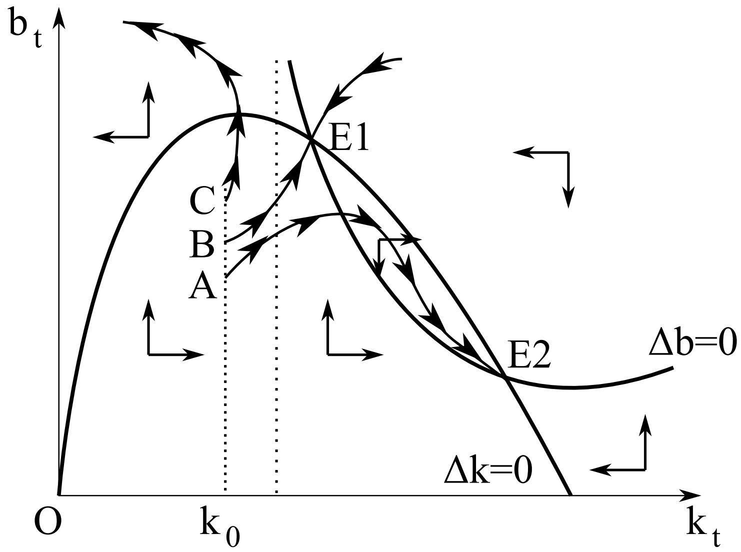

Figure 1: A Phase Diagram (A Case of Sufficiently Small D).

a *balanced growth path steady state (steady state, henceforth)* if they constitute a competitive equilibrium and if the predetermined variables grow at the same rate of $1 + n$ for any t.

In the overlapping generations economy considered here, it is shown that there exists a critical value regarding the ratio of primary deficit to GDP such that, if the ratio of primary deficit to GDP, D, is lower than the critical value, there exist two steady states. Meanwhile, if the ratio of primary deficit to GDP, D, is higher than the critical value, there exists no steady state. In order to show the properties, we use phase diagrams with respect to the levels of public debt per capita and of physical capital per capita.

Figure 1 illustrates the dynamics of the levels of physical capital per capita, k_t, and of public debt per capita, b_t, when the ratio of primary deficit to GDP, D, is smaller than the critical value. In figure 1, there exist two steady states, E1 and E2. We obtain that if the initial level of public debt is low relative to that of physical capital, the economy converges to the locally-stable steady state, E2.

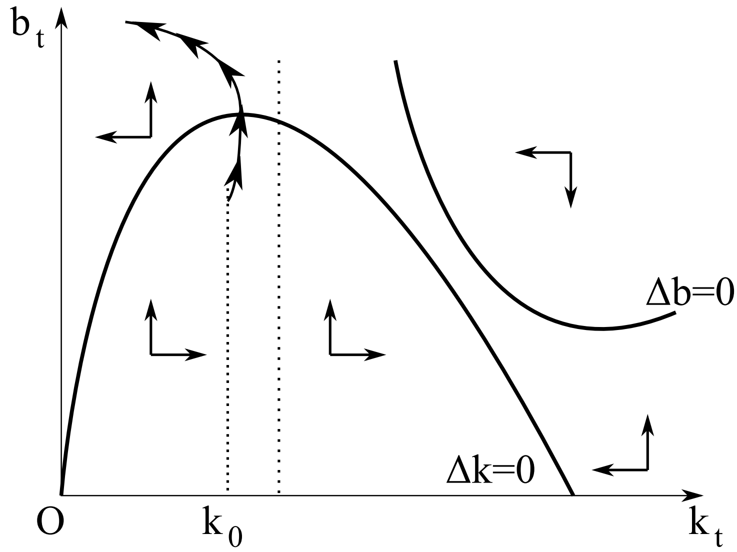

Figure 2: A Phase Diagram (A Case of Too Large D).

Meanwhile, if the initial level of public debt is high relative to that of physical capital, the ratio of public debt to GDP diverges and the given primary deficit to GDP ratio is not sustainable because physical capital is crowded out by the rapid accumulation of public debt[8]. For instance, suppose that an initial level of physical capital is given by k_0 in figure 1. If the initial level of public debt is smaller than the level at point B (for example, let the initial point be A), this economy converges to E2, which is the locally-stable steady state. Meanwhile, if the initial level of public debt is larger than the level at point B (for example, let the initial point be C), public debt continues to accumulate and the physical capital will be crowded out, which leads to the divergence of the ratio of public debt to GDP.

However, if the ratio of primary deficit to GDP is higher than the critical value, the properties of the dynamics change, which is illustrated in figure

[8]In this case, the level of public debt per capita grows and that of physical capital per capita decreases to zero in finite time. Therefore, the ratio of public debt to GDP increases infinitely in finite time.

2. In this case, there exists no steady state: starting from any level of initial public debt, the ratio of public debt to GDP diverges infinitely. In other words, regardless of the level of initial public debt, the constant primary deficit to GDP ratio cannot be sustained and the government must improve its fiscal balance.

In sum, we obtain the following two results. First, if the ratio of primary deficit to GDP is lower than the critical value of the ratio, there exists a steady state. Furthermore, if the initial amount of public debt is also sufficiently small, the ratio of public debt to GDP will converge to some finite level. Second, if the ratio of primary deficit to GDP is higher than the critical value, there exists no steady state. In this case, the ratio of public debt to GDP diverges infinitely, whatever the initial amount of public debt.

3 The Overlapping Generations Model with Multiple Generations

In this section, we construct an overlapping generations model to find the maximum sustainable level of the primary deficit. Unlike the model explained in the previous section, we assume that households live for multiple periods in order to calibrate model parameters to match yearly data for the Japanese economy. We explain the outline of the overlapping generations model used in our numerical evaluations; the details are given in the appendix.

3.1 Households

In every period, households are born and live for T periods. The population of households born in period t is denoted by $N(t)$, which grows at the rate of $1+n$, that is, $N(t+1) = (1+n)N(t)$. Households obtain utility from consumption and leisure. A periodic utility function is given by

$$u(c_t(t+j), l_t(t+j)) = \frac{1}{1-\gamma}[c_t(t+j)^{\theta}(1-l_t(t+j))^{1-\theta}]^{1-\gamma}, \tag{8}$$

where $j = 0, 1, \cdots, T$ is their age, and $c_t(t+j)$ and $1 - l_t(t+j)$ are the amounts of consumption and leisure of households born in period t when their age is j, respectively. γ and θ are preference parameters. Households face flow budget constraints as follows:

$$c_t(t+j) + z_t(t+j) = w(t+j)e_t(t+j)l_t(t+j) + R(t+j)z_t(t+j-1), \tag{9}$$

$$l_t(t+j) \geq 0, \tag{10}$$

where $z_t(t+j)$ is the asset holdings of generation t in period $t+j$. $\{e_t(t+j)\}_{j=0}^{T}$ represents the profiles of labor productivity, which varies by their generation and age. In this paper, we assume that $e_{t+1}(t+1+j) = (1+h)e_t(t+j)$ for any t and j. h is the growth rate of labor productivity and is exogenously given. Households maximize their lifetime utility, which is defined as

$$\sum_{j=0}^{T} \beta^j u(c_t(t+j), l_t(t+j)) \tag{11}$$

subject to their lifetime budget constraints.

3.2 Firms

A representative firm produces final goods from labor and physical capital. The final goods market is perfectly competitive. A firm's production technology is represented by a Cobb–Douglas production function as $y(t) = f(k(t)) = Ak(t)^{\alpha}$, where $y(t)$ and $k(t)$ are output per capita and capital per capita, respectively. A and α are exogenous parameters. Thus, the firm's profit maximization conditions are as follows:

$$r(t) = A\alpha k(t)^{\alpha-1}, \tag{12}$$

$$w(t) = A(1-\alpha)k(t)^{1-\alpha}. \tag{13}$$

Lastly, δ is the depreciation rate of physical capital.

3.3 Government

The government is assumed to maintain a constant ratio of primary deficit, $D(t)$, to GDP, $Y(t)$: D is constant for all t, where $D = D(t)/Y(t)$. The deficit is a wasteful one and thus does not directly affect the economy[9]. The primary deficit and the interest payment on public debt are financed by the issuance of public debt. The government has to conform to the flow budget constraints in every period,

$$B(t+1) = R(t)B(t) + D(t), \tag{14}$$

where $B(t)$ is an amount of public debt in period t and $R(t)$ is the interest rate on public debt.

3.4 Competitive Equilibrium

We define a competitive equilibrium as follows.

Definition 1 (Competitive equilibrium)**.** *Given a constant ratio of public deficit to GDP, D, given initial amounts of physical capital, K_0, and public debt, B_0, a set of sequences of state variables $\{K(t+1), B(t+1)\}_{t=0}^{\infty}$, allocations $\{\{c_t(t+j)\}_{j=0}^{T}, \{l_t(t+j)\}_{j=0}^{T}\}_{t=0}^{\infty}$, and a price system $\{R(t), w(t), r(t)\}_{t=0}^{\infty}$ is a competitive equilibrium if, for all t, the sequences satisfy the following conditions:*

[9]We take the primary deficit as the difference between tax revenues and government spending excluding government transfer payments. In Section 5.1, we introduce components of tax revenues and government transfer payments into our analysis.

1. *Given the price system, the allocations maximize households' lifetime utility subject to their lifetime budget constraints;*

2. *given the price system, the allocations and state variables maximize firm's profit;*

3. *they satisfy the government's flow budget constraints and the constant primary deficit per GDP rule;*

4. *they satisfy the no-arbitrage condition,* $R(t) = 1 + r(t) - \delta$*; and*

5. *they clear all markets.*

In a competitive equilibrium, the dynamical system can be transcribed [10] as

$$Z({}_{t-T}R^{t+T}) = K(t+1) + B(t+1), \tag{15}$$

$$B(t+1) = R(t)B(t) + D(t) \tag{16}$$

where ${}_{t-T}R^{t+T} = \{R(t-T), R(t-T+1), \cdots, R(t+T-1), R(t+T)\}$ indicates the sequence of interest rates of public debt from period $t-T$ to $t+T$, and $Z({}_{t-T}R^{t+T})$ is aggregate asset holdings in period t.

3.5 Balanced Growth Path Steady State

We focus on a balanced growth path steady state in order to investigate the types of fiscal policies that are sustainable. A balanced growth path steady state is defined as follows.

Definition 2. *A set of sequences of state variables* $\{K(t+1), B(t+1)\}_{t=0}^{\infty}$*, allocations* $\{\{c_t(t+j)\}_{j=0}^{T}, \{l_t(t+j)\}_{j=0}^{T}\}_{t=0}^{\infty}$ *and a price system* $\{R(t), w(t), r(t)\}_{t=0}^{\infty}$ *is a balanced growth path steady state ("steady state") if the set is a competitive equilibrium and the interest rate,* $R(t)$*, is constant for all* t*,* $R(t) = R$*.*

[10] The derivation of the dynamical system, equations (15) and (16), is demonstrated in the appendix.

If there exists a steady state, the following conditions are satisfied in the steady state:

$$Z(t,R) = K(t+1,R) + B(t+1,R), \tag{17}$$

$$B(t+1,R) = RB(t,R) + D(t,R). \tag{18}$$

Eliminating B, we obtain

$$\Phi(R) = \Theta(R;D). \tag{19}$$

The derivation of equation (19) is explained in the appendix. However, we can intuitively understand equation (19). $\Phi(R)$ corresponds to the ratio of aggregate asset holdings to GDP in the steady state. $\Theta(R;D)$ is the sum of the levels of the physical capital-to-GDP ratio and the public debt-to-GDP ratio.

We can show that there is a critical value of primary deficit $D^* \geq 0$ such that

- if $D \leq D^*$, equation (19) has a solution with respect to R, and
- if $D > D^*$, equation (19) has no solution with respect to R.

In other words, when the size of the primary deficit is too large, the dynamical system has no steady state. Therefore, we can consider that there exists a maximum sustainable level of primary deficit per GDP. Furthermore, the theoretical results obtained in the previous section imply that if there is no steady state, the ratio of public debt to GDP necessarily diverges infinitely for *any* initial amount of public debt. In the following section, we calibrate the maximum sustainable size of the primary deficit per GDP and find the necessary condition for preventing the ratio of public debt to GDP from diverging.

4 Numerical Evaluation

We numerically evaluate the maximum sustainable level of the primary deficit in this section. We take three steps to obtain quantitative findings using the model introduced in the previous section. First, we calibrate the preference parameters, β, θ, and γ, using data around 1985 to consider the steady state of the Japanese economy. Second, we calibrate the other parameters using the calibrated preference parameters and data around 2005 to take into account the recent economic environment. Lastly, we numerically calculate the sustainable size of the primary deficit-to-GDP ratio subject to different labor productivity growth rates and determine the condition under which Japan's government can run a primary deficit.

4.1 Calibrating Preference Parameters

We calibrate the preference parameters using data for the Japanese economy around 1985. We focus on this period because the relationship between the size of the primary fiscal balance and the amount of public debt can be regarded as a steady state at that time. The average ratio of Japan's primary surplus to GDP was 0.465% and that of the net financial liabilities of Japan's general government to GDP was 29.9% in 1983–1987[11]. This relationship can be sustained in the long run. Therefore, we calibrate the preference parameters using the data in 1983–1987 under the assumption that the Japanese economy and fiscal position were in a steady state.

We set the model parameters to calibrate the preference parameters as follows. The length of a lifetime, T, is 61, which means that households live for 62 periods[12]. The weight parameter in the production function, α, is set to

[11] The data used in the calibration are calendar year (CY) ones.

[12] We suppose that individuals enter the economy when they are 20 years old. Thus, households live from 20 to 81 years.

0.3516 to match the average ratio of capital income to GDP in 1983–1987[13]. The growth rates of population, n, and labor productivity, h, are set to $n = 0$ and $h = 0.0175$, respectively, in the steady state[14]. The profiles of households' labor productivity depend on their generation and age. Following Ishikawa et al. (2012), we estimate the profiles of labor productivity as follows.

$$\frac{\exp(0.691591 + 0.044425 \cdot j - 0.00086 \cdot j^2)}{\exp(0.691591)} \times (1+h)^{t-1}, \tag{20}$$

where j is a household's age ($j = 0, 1, \cdots, T$) and h is the growth rate of labor productivity. Lastly, the intertemporal elasticity of substitution is assumed to be 0.8.

Using our model and the parameters calibrated above, the preference parameters, β, θ, and γ, are also calibrated by matching the average data for the Japanese economy in 1983–1987. The target ratios of capital to GDP and public debt to GDP in the steady state are set to $K/Y = 1.9346$ and $B/Y = 0.2993$, respectively[15]. The target ratio of the average working time to the discretionary time of households is given as 0.5767[16]. Lastly, the target real interest rate on public debt is set to $R - 1 = 0.04376$, which is consistent with the average of Japan's nominal interest rates on public debt, 6.370%, and the average of inflation rates of Japan, 1.911%, in 1983–1987[17]. We obtain the calibrated parameters,

[13]*Capital income* is the sum of (i) operating surplus, (ii) 40% of mixed income, and (iii) consumption of fixed capital. These data are obtained from Japan's SNA national accounts data.

[14]The growth rates set here, h and n, are lower than the average growth rates in 1983–1987. We consider that the actual growth rates cannot be kept in the long run and thus, the steady state growth rates must be lower.

[15]These are obtained from SNA national accounts data. The amount of physical capital, K, is determined in the same way as in Hayashi and Prescott (2002). The amount of public debt, B, is the net financial liabilities of Japan's general government.

[16]The target ratio is set in accordance with 1986 data from the *Survey of Time Use and Leisure Activities* by the Ministry of Internal Affairs and Communications.

[17]The nominal interest rates are obtained as the average of the interest rates of Japanese government bonds whose remaining duration was nine years in each year from 1983 to 1987. The longest remaining duration is nine years in the available interest rates reported in the dataset from Japan's Ministry of Finance, and thus, we take the average as the long-run interest rate. The inflation rate is made from the GDP deflators.

$\beta = 0.9676$, $\theta = 0.6564$, and $\gamma = 1.381$, to maintain consistency with the data discussed above.

4.2 Calibrating the Remaining Parameters

Next, we calibrate the other parameters reflecting recent Japan's economic situation, by using recent data for the Japanese economy and the preference parameters obtained in the previous subsection.

The length of a lifetime, T, remains 61. The parameter in the production function is set to $\alpha = 0.3748$ based on the average of the capital income share in 2003–2007. The depreciation rate of physical capital is calibrated as $\delta = 0.0902$ based on the average ratio of the size of consumption of fixed capital to the amount of physical capital in 2003–2007[18]. The profiles of labor productivity are estimated in the same way as in Section 4.1, and we obtain

$$\frac{\exp(0.971 + 0.0485 \cdot j - 0.000915 \cdot j^2)}{\exp(0.971)} \times (1+h)^{t-1}, \tag{21}$$

where j is a household's age. The growth rate of population is assumed to be zero[19]. We note that the growth rate of labor productivity, h, is given afterward.

In addition, we adopt an exogenous difference between the interest rate on public debt and the return on physical capital in order to replicate the actual gap between them; thus, the no-arbitrage condition is rewritten as

$$R(t) = 1 + r(t) - \delta - s. \tag{22}$$

Sakuragawa and Hosono (2010) emphasize the importance of the difference in

[18] The size of consumption of fixed capital is obtained from the SNA national accounts data. The definition of physical capital follows that of Hayashi and Prescott (2002), and we calculate the amount of physical capital in 2003–2007.

[19] In our model, the population growth has the same effect as the labor productivity growth, and we can easily analyze the impact of population growth (e.g., an increase of immigrations by relaxing Japan's immigration policy) by increasing h.

T	length of lifetime	61
α	capital income share	0.3748
δ	depreciation rate of physical capital	0.0902
s	difference between the interest rates	0.0360
h	growth rate of labor productivity	(exogenously given)
n	growth rate of population	0.000
$e_t(t+j)$	profile of labor productivity	$\frac{\exp(0.971 + 0.0485 \cdot j - 0.000915 \cdot j^2)}{\exp(0.971)} \times (1+h)^{t-1}$
γ	parameter corresponding to RRA	1.381
θ	weight parameter between c and $1-l$	0.6564
β	subjective discount rate	0.9676

Table 1: List of Calibrated Parameters

analysis of fiscal sustainability. Ueda (2012) explains that the interest rates on public debt in Japan have been around 1.5% since 2000, while the rates of return on physical capital have been 4.5–5.0%[20]. In our paper, the difference between the rates is set to $s = 0.036$, which means that the interest rate on public debt continues to be lower than the rate of return on capital by 3.6%[21].

Table 1 is the list of the values of the calibrated model parameters, and we define a *benchmark case* as one using them.

4.3 Results of Numerical Calculations

In order to determine the necessary condition for a sustainable primary deficit, we exogenously give various growth rates of labor productivity, h, and then calculate the maximum sustainable level of the ratio of the primary deficit to GDP.

The results of our numerical calculations are tabulated in Table 2. We show

[20]There are several reasons why the interest rates on public debt tend to be lower than the rate of return on physical capital: contributing factors include risk premiums for volatile revenue, financial intermediation costs, and uncertainty about tail risk of devastating disaster.

[21]Although the difference between the rates is exogenously given from the data in our study, we should investigate effects of *endogenous* interest gap on sustainable fiscal policies. We find that replacing the exogenous interest gap with an endogenous one does not change our main result and implication insofar as we investigate it. The details of our analysis are available upon request.

h	Maximum P.D. (%)	Debt Interest Rate (%)	K/Y	B/Y
$h = 0.055$	N.A.	—	—	—
$h = 0.060$	0.02102	5.8685	2.0269	0.1599
$h = 0.075$	0.4634	6.8323	1.9265	0.6941
$h = 0.080$	0.7039	7.1565	1.8949	0.8345
$h = 0.100$	1.9131	8.4572	1.7779	1.2400

Table 2: Maximum Sustainable Levels of Primary Deficit per GDP

Note: h is the growth rate of labor productivity and "Maximum P.D." is the maximum sustainable level of the ratio of primary deficit to GDP (%). "N.A." means that there exists no steady state for any levels of the primary deficit per GDP. "Debt Interest Rate," K/Y, and B/Y are the real interest rate on public debt (%), the ratio of capital to GDP, and the public debt per GDP in the steady state, respectively.

that the economic growth rate needs to be more than 5.5% to sustain primary deficits, since under $h = 0.055$ we could not find a solution that satisfies the condition for a steady state. Even if the economic growth rate is assumed to be 8%, the sustainable primary deficit is only 0.78% of GDP. We note that the conditions for the size of the primary deficit are necessary but not sufficient to guarantee fiscal sustainability. As shown in Chalk (2000), the ratio of public debt to GDP does not diverge if and only if there exists a steady state *and* the initial amount of public debt is sufficiently small. If the initial amount of public debt is sufficiently large and that of physical capital is sufficiently small, accumulation of physical capital continues to be crowded out, and it is impossible to prevent the ratio of public debt to GDP from diverging. Even if the growth rate might continue to be higher than the interest rate, it would take a very long time to reduce the debt-to-GDP ratio to the steady-state level with the maximum primary deficit under the condition that the initial debt-to-GDP ratio is high.

Our results imply that a primary surplus is necessary for Japan in the long run in order to avoid divergence of the public debt-to-GDP ratio under realistic assumptions of the economic growth rate. In other words, it is unrealistic to expect to attain fiscal sustainability only by boosting economic growth without eliminating the primary deficit in Japan.

Compared to the result of Chalk (2000), our investigation leads to the severe situation of Japan's fiscal sustainability[22]. The difference between the results arises from the differences in the calibrated model parameters, particularly the preference parameters, β and γ. While we set $\beta = 0.9676$ and $\gamma = 1.381$ in the benchmark case, Chalk (2000) calibrates $\beta = 0.99$ and $\gamma = 2.25$. Individuals have more incentive to hold assets in Chalk (2000), and thus, a larger primary deficit tends to be sustainable in his analysis. The reason β and γ are calibrated in this manner is that Chalk (2000) supposes the 1954–1980 situation, under which the average fiscal deficit is zero and U.S. (government bond) interest rates are much lower than the economic growth rates[23]. Of course, other parameters are also important for evaluating sustainable fiscal policies, for example, economic growth rates, lifetime of households, and interest rate gap. However, β and γ have substantial effects on the maximum size of the sustainable primary deficit.

4.4 Comparative Analysis

Next, we show the result of a comparative analysis of the intertemporal elasticity of substitution. Household intertemporal elasticity of substitution in Japan has been estimated in many previous studies, such as Hamori (1996) and Fuse (2004). Because the estimated values are so diverse and the intertemporal elasticity of substitution may substantially affect the maximum size of the sustainable primary deficit, as explained in Section 4.1, we have to verify the robustness of our result based on various values of the intertemporal elasticity

[22] In the benchmark case of Chalk (2000), under 3.0% economic growth rate (1.7% population growth plus 1.3% productivity growth), the maximum size of primary deficit is calculated as 5.1% of GDP.

[23] In Chalk (2000), parameters are calibrated under the assumptions that the interest rate is 1.2% and the economic growth rate is 3.0%. The assumptions we adopt are explained in Section 4.1.

IES (target)	θ	γ	β
IES = 0.3	0.6490	4.5953	0.9919
IES = 0.5	0.6517	2.5344	0.9763
IES = 0.8 (benchmark)	0.6564	1.3809	0.9676
IES = 1.2	0.6382	0.7388	0.9626

Table 3: List of Alternative Preference Parameters under Various IES

Note: "IES" is the intertemporal elasticity of substitution. This table describes the calibrated values of preference parameters, θ, γ, and β, to match to the target level of IES using 1985 data for the Japanese economy.

	$h = 0.05$	$h = 0.06$	$h = 0.08$	$h = 0.10$
IES = 0.3	N.A.	N.A.	N.A.	N.A.
IES = 0.5	N.A.	N.A.	8.653×10^{-5}	0.06808
IES = 0.8 (benchmark)	N.A.	0.02102	0.7039	1.9131
IES = 1.2	N.A.	0.4007	2.998	6.708

Table 4: Maximum Sustainable Levels of Primary Deficit under Various IES

Note: "IES" and h are the intertemporal elasticity of substitution and the growth rate of labor productivity, respectively. This table describes the maximum sustainable levels of the ratio of the primary deficit to GDP (%). "N.A." means that there exists no steady state for any level of the primary deficit to GDP.

of substitution.

The procedure of our comparative analysis is as follows. First, we assume another value of the intertemporal elasticity of substitution. Second, we again calibrate the preference parameters, β, γ, and θ, using the data for the Japanese economy around 1985. Third, we calibrate the other parameters to match the data around 2005 and recalculate the maximum sustainable level of the primary deficit with various growth rates. We consider three cases, in which the values of the intertemporal elasticity of substitution are assumed to be 0.3, 0.5, and 1.2[24], and obtain the calibrated preference parameters shown in Table 3.

Using the preference parameters, we calculate the maximum sustainable level of the primary deficit under various growth rates. Our results in Table 4

[24] We note that if we assume too large a value of the intertemporal elasticity of substitution, there is no preference parameter to match the data for the Japanese economy in 1985.

show that the higher (lower) intertemporal elasticity of substitution leads to a larger (smaller) sustainable primary deficit. This means that under the higher intertemporal elasticity of substitution, households facing a higher interest rate would like to increase their labor supply and asset holdings because of a stronger substitution effect, which increases the demand for public debt and lowers the interest rate.

Even if the intertemporal elasticity of substitution is set to 1.2, however, we would still need a 5% economic growth rate in order for Japan's government to maintain the primary deficits.

5 Extensions

In this section, we elaborate on extensions of our model in two directions: we introduce tax systems and transfers as well as introducing stochastic survival rates. Thus far, we have focused on the magnitude of sustainable primary deficits under the assumption that the primary deficit is defined by wasteful government expenditure and taxes levied by non-distortionary instruments. In order to apply the analysis implication to actual policies, we should consider the composition of a primary deficit and the effects of distortion by tax instruments. Considering actual tax systems is important because the magnitude of tax distortion differs among various tax instruments, and the size of sustainable government expenditure changes under different tax systems. We also consider stochastic survival rates and realistic saving behaviors of households. Stochastic survival rates affect the amount of aggregate asset holdings and may substantially affect sustainable fiscal policies, although the stochastic survival rates do not change our main result and implication obtained in the previous section.

5.1 Introducing Tax Systems

In this subsection, we introduce three tax instruments, labor income, capital income, and consumption taxation, into our calibrated model and investigate how large the effects of tax distortions are at the steady state on the sustainable levels of government expenditure. Each tax instrument has a distortion: labor income and consumption taxation distort individuals' labor–leisure choice and capital income taxation distorts the intertemporal choice. If the tax distortions are small, the introduction of the tax systems substantially increases the maximum sustainable government expenditure and does not reduce the sustainable size of the primary deficit. If the tax distortions are large, conversely, the maximum sustainable primary deficit declines by the introduction of the tax systems. We also consider the actual magnitude of transfers from the government to households in order to determine the sustainable level of wasteful government expenditure.

The procedure of our investigation is as follows. First, we give and fix the three tax rates, τ^L, τ^K, and τ^C, which are exogenous and time-invariant. The labor income tax rate τ^L is set to 26.3% as in Ishikawa et al. (2012). The capital income tax rate τ^K is set to 30.0%, and the consumption tax rate is set to 5%, corresponding to the rates in the current Japanese economy. Next, we assume that households aged 45 (corresponding to 65 years old) or over receive a constant transfer $w(1+h)^{-i}p$ until they exit the economy, where w is the wage rate, h is the growth rate of productivity, i is the generation of households, and p is a constant value[25]. p is calibrated to match the 2005 data for the Japanese economy and we set $p = 3.987$[26]. Lastly, we assume that other government

[25] In this extension, we consider public pension payments only as government transfer payments because the pension payments account for a large proportion of government transfer payments. The setting of the amount of transfer reflects that size of pension benefits depend on households' labor incomes, which is affected by the wage rate w and the profile of labor productivity.

[26] We calibrate p to match the following three conditions. First, the government transfer relative to GDP is set to 8.49% to match the average of government pension benefits-to-GDP ratio in 2003–

τ^L	τ^K	τ^C	Max.G.S. (%)	G.T. (%)	T.R. (%)	P.D. (%)
0	0	0	−2.882	6.680	0	3.797
0.263	0	0.050	15.69	9.356	19.06	5.985
0	0.300	0	3.069	6.686	10.55	−0.7934
0.263	0.300	0.050	21.60	9.365	28.31	2.657

Table 5: Maximum Sustainable Levels of Primary Deficit per GDP

Note: τ^L, τ^K, and τ^C are the labor income, capital income, and consumption tax rates, respectively. These tax rates are constant and time-invariant. "Max.G.S." means the maximum sustainable level of the ratio of government spending to GDP (%). "G.T." and "T.R." mean the ratio of total government transfer to GDP and that of total tax revenue to GDP (%) in the steady state, respectively. "P.D." means the primary deficit-to-GDP ratio (%) in the steady state, which is calculated as "Max.G.S." minus "T.R." In all cases, the economic growth rate is assumed to be 10.0%, and the transfer per capita (to households aged 45 or over) is given by $3.987 \times w(1+h)^{-i}$.

(wasteful) spending relative to GDP is constant for all t, $\tilde{G} = G_t/Y_t$. Then, we numerically derive the maximum size of the other government spending to GDP under which there exists a steady state. If $\tilde{G}$ is larger than the threshold level, there is no steady state and the ratio of public debt to GDP will diverge regardless of the initial conditions on public debt and physical capital. We note that when all the tax rates and the size of the transfers are zero, this case goes back to the benchmark one.

The results are reported in Table 5, which gives us the implications that capital income taxation largely distorts individuals' intertemporal choice and that the maximum levels of sustainable primary deficit per GDP tend to substantially decline. Meanwhile, the labor income and consumption taxation have only small distortion effects in the steady state, and the introduction of labor income and consumption taxation does not decrease the maximum size of the primary deficit relative to GDP.

2007, which is obtained by Japan's SNA national account data. Second, the household working time of each generation is set using the 2006 data of the *Survey of Time Use and Leisure Activities* by the Ministry of Internal Affairs and Communications. Third, the profile of labor productivity is given by equation (21).

5.2 Introducing Stochastic Survival Rates

In this subsection, we introduce stochastic survival rates into our model. Chen et al. (2007) and Braun et al. (2009) project Japan's national saving rate using computable overlapping generations models with individuals' stochastic survival rates. In their model, the survival rates give more precise demographic dynamics and affect the saving behavior of households. We will show that an introduction of stochastic survival rates may have a sizable effect on sustainable fiscal policies.

The stochastic survival rates conditioned on an individual's age are represented by $\psi(i)$, where i is the individual's age. $\psi(i)$ means that an individual whose age is i in period t survives in period $t+1$ with probability $\psi(i)$. Given $\{\psi(i)\}_{i=0}^{T-1}$, we define the unconditional stochastic survival rates $s(i)$ as

$$s(0) = 1, s(i) = \psi(i)s(i-1) \text{ for all } i = 1, 2, \cdots, T. \tag{23}$$

Thus, individuals' lifetime utility is given by

$$\sum_{j=0}^{T} s(j)\beta^j u(c_t(t+j), l_t(t+j)), \tag{24}$$

and individuals face the (flow) budget constraints given by

$$c_t(t+j) + z_t(t+j) = w(t+j)e_t(t+j)l_t(t+j) + R(t+j)z_t(t+j-1) + \xi(t+j) \tag{25}$$

for all t and j, where $\xi(t+j)$ is a lump-sum transfer from the government. The lump-sum transfer is financed by aggregate accidental bequests. ξ is

Survival Rates	Maximum P.D. (%)	Debt Interest Rate (%)	K/Y	B/Y
No (Benchmark)	0.7361	6.6715	1.9245	0.8884
Yes	0.2943	6.9754	1.8949	0.5610

Table 6: Maximum Sustainable Levels of Primary Deficit per GDP with the Stochastic Survival Rates

Note: h is the growth rate of labor productivity and "Maximum P.D." is the maximum sustainable level of the ratio of primary deficit to GDP (%). "Debt Interest Rate," K/Y, and B/Y are the real interest rate on public debt (%), the ratio of capital to GDP, and the public debt per GDP in the steady state, respectively. In all cases, the economic growth rate is assumed to be 7.5 %.

determined by the following equation:

$$\sum_{j=0}^{T} \xi(t+1)s(j)N(t+1-j) = \sum_{j=0}^{T-1} R(t+1)z_{t-j}(t)(1-\psi(j))s(j)N(t-j). \tag{26}$$

The right-hand side of equation (26) represents the aggregate accidental bequests in the beginning of period $t+1$, while the left-hand side of equation (26) represents the total lump-sum transfer in period $t+1$.

We calibrate the conditional stochastic survival rates as follows[27]:

$$\psi(i) = 1 - \text{Deaths}(i)/\text{Population}(i). \tag{27}$$

Deaths(i) is reported in 2005 *Vital Statistics* by the Ministry of Health, Labor and Welfare. In order to determine Population(i), we use data from the 2005 *Annual Report on Current Population Estimates* by the Ministry of Internal Affairs and Communications.

We show the result under the assumption that the growth rate of labor productivity is 7.5% in Table 6. Introducing the stochastic survival rates decreases the size of the maximum sustainable primary deficit relative to GDP from 0.7361% to 0.2943%. It is shown that the (stochastic) survival rates lower

[27] This method of calibrating $\psi(i)$ is also used in Braun et al. (2009).

the benefit from saving and have a negative effect on the aggregate saving of households.

6 Conclusion

We investigated how large a primary deficit-to-GDP ratio Japan's government can sustain. We constructed an overlapping generations model in which multi-generational households live and the government maintains a constant ratio of the primary deficit to GDP, and showed that the primary deficit cannot be sustained unless the rate of economic growth is unrealistically high. Our result implies that Japan's government needs to achieve a positive primary balance in order to avoid the divergence of the ratio of public debt to GDP.

However, most some problems with our analysis remain to be solved. One of the most important is to obtain the sustainable size of initial public debt. As shown by Chalk (2000), a sufficiently small amount of initial public debt is also necessary to avoid the infinite divergence of the ratio of public debt to GDP. Nevertheless, we have not investigated the conditions required because the calculation of the transition paths of public debt and physical capital are complex. Another problem is the introduction of the imperfections of financial markets and the systems of public pension and health care spending in detail in order to incorporate the real frictions and actual policies in the analysis of fiscal sustainability.

Although these problems remain, we consider our research to be valuable for suggesting future directions for Japan's fiscal stance. We believe that this paper contributes not only to the fields of macroeconomic and public finance literature, but also to determining on future fiscal policies for Japan.

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Appendix

A Details of Overlapping Generations Model with Multi Generations

A.1 Household's Problem

The utility maximization problem of household born at period t is given by:

$$\max \sum_{i=0}^{T} \beta^i \frac{[c_t(t+i)^{\theta}(1-l_t(t+i))^{1-\theta}]^{1-\gamma}}{1-\gamma} \tag{28}$$

$$\begin{aligned}
\text{s.t. } & c_t(t) + z_t(t) = w(t)e_t(t)l_t(t) \\
& c_t(t+1) + z_t(t+1) = w(t+1)e_t(t+1)l_t(t+1) + R(t+1)z_t(t) \\
& \vdots \\
& c_t(T) = w(T)e_t(T)l_t(T) + R(T)z_t(T-1), \\
& l_t(t+i) \geq 0 \;\; \forall i = 0, 1, \cdots, T.
\end{aligned} \tag{29}$$

Combining the flow-budget constraints, we obtain the lifetime budget constraint. To solve the problem, we define the Lagrangian as:

$$\begin{aligned}
\mathcal{L} = & \sum_{i=0}^{T} \beta^i \frac{[c_t(t+i)^{\theta}(1-l_t(t+i))^{1-\theta}]^{1-\gamma}}{1-\gamma} \\
+ & \mu \left[\sum_{i=0}^{T} w(t+i)e_t(t+i)l_t(t+i) \times \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) - \sum_{i=0}^{T} c_t(t+i) \times \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) \right] + \sum_{i=0}^{T} \lambda_i l_t(t+i).
\end{aligned} \tag{30}$$

The first-order conditions are

$$\beta^i \theta c_t(t+i)^{\theta-1}(1-l_t(t+i))^{1-\theta} \times [c_t(t+i)^{\theta}(1-l_t(t+i))^{1-\theta}]^{-\gamma} - \mu \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) = 0 \tag{31}$$

$$\Rightarrow \beta^i \theta c_t(t+i)^{\theta(1-\gamma)-1}(1-l_t(t+i))^{(1-\theta)(1-\gamma)} - \mu \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) = 0, \tag{32}$$

$$-\beta^i (1-\theta) c_t(t+i)^{\theta}(1-l_t(t+i))^{-\theta} \times [c_t(t+i)^{\theta}(1-l_t(t+i))^{1-\theta}]^{-\gamma} + \mu w(t+i) e_t(t+i) \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) + \lambda_i = 0 \tag{33}$$

$$\Rightarrow \beta^i (1-\theta) c_t(t+i)^{\theta(1-\gamma)}(1-l_t(t+i))^{(1-\theta)(1-\gamma)-1} - \mu w(t+i) e_t(t+i) \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) - \lambda_i = 0. \tag{34}$$

If $\lambda_i = 0$, we obtain the following equations from the two first-order conditions:

$$c_t(t+i) = \left(\frac{\theta}{\mu} \right)^{1/\gamma} \left(\prod_{j=1}^{i} (\beta R(t+j)) \right)^{1/\gamma} \left[w(t+i) e_t(t+i) \frac{\theta}{1-\theta} \right]^{-(1-\theta)(1-\gamma)/\gamma}, \tag{35}$$

$$\frac{1-\theta}{\theta} c_t(t+i) = (1-l_t(t+i)) w(t+i) e_t(t+i). \tag{36}$$

Meanwhile, if $\lambda_i > 0$, we have $l_t(t+i) = 0$ from the complementarity condition. Combining the first order condition and $l_t(t+i) = 0$,

$$c_t(t+i) = \left[\frac{\theta}{\mu} \left(\prod_{j=1}^{i} (\beta R(t+j)) \right) \right]^{\frac{1}{(1-\theta)(1-\gamma)+\gamma}} \tag{37}$$

$$\beta^i (1-\theta) c_t(t+i)^{\theta(1-\gamma)} - \mu w(t+i) e_t(t+i) \left(\prod_{j=1}^{i} R(t+j)^{-1} \right) > 0 \tag{38}$$

Then, given $\{e_t(t+i)\}_{i=0}^{T}$ and $\{w(t+i), R(t+i)\}_{i=0}^{T}$,

- if $\mu \geq \tilde{\mu}(i)$,

$$c_t(t+i) = \left(\frac{\theta}{\mu}\right)^{1/\gamma} \left(\prod_{j=1}^{i} (\beta R(t+j))\right)^{1/\gamma} \left[w(t+i)e_t(t+i)\frac{\theta}{1-\theta}\right]^{-(1-\theta)(1-\gamma)/\gamma}, \tag{39}$$

$$l_t(t+i) = 1 - \frac{1-\theta}{\theta}\frac{c_t(t+i)}{w(t+i)e_t(t+i)}, \tag{40}$$

- and if $\mu < \tilde{\mu}(i)$,

$$c_t(t+i) = \left[\frac{\theta}{\mu}\left(\prod_{j=1}^{i}(\beta R(t+j))\right)\right]^{\frac{1}{(1-\theta)(1-\gamma)+\gamma}}, \tag{41}$$

$$l_t(t+i) = 0, \tag{42}$$

where $\tilde{\mu}(i)$ is defined as

$$\tilde{\mu}(i) = [w(t+i)e_t(t+i)]^{-(1-\theta)(1-\gamma)-\gamma}\, \theta^{\theta(1-\gamma)}(1-\theta)^{(1-\theta)(1-\gamma)+\gamma}\left(\prod_{j=1}^{i}(\beta R(t+j))\right). \tag{43}$$

To derive the profiles of consumption and leisure of households, we need to obtain the value of μ. μ can be obtained by substituting equations (39)-(42) into the household's intertemporal budget constraint.

From the profiles of consumption and leisure of household $\{c_t(t+i), l_t(t+i)\}_{i=0}^{T}$, we can recursively calculate the profile of asset holdings $\{z_t(t+i)\}_{i=0}^{T}$ using the flow budget constraints.

Finally, we derive the aggregate asset holdings $Z(t)$ as

$$Z(t) = \sum_{i=0}^{T} z_{t-i}(t)N_{t-i} = N(1)\sum_{i=0}^{T} z_{t-i}(t)(1+n)^{t-i-1}. \tag{44}$$

A.2 Firm's Problem

The firm solves the following profit-maximization problem.

$$\max A(t)K(t)^{\alpha}L(t)^{1-\alpha} - r(t)K(t) - w(t)L(t). \tag{45}$$

where $L(t) = \sum_{i=0}^{T} e_{t-i}(t)l_{t-i}(t)N_{t-i}$. F.O.C.s are

$$r(t) = A(t)\alpha K(t)^{\alpha-1}L(t)^{1-\alpha} \tag{46}$$

$$w(t) = A(t)(1-\alpha)K(t)^{\alpha}L(t)^{-\alpha} \tag{47}$$

A.3 Government

A Government finances the primary deficit, $D(t)$, and the rollover plus the interest payment of the existing public debt, $R(t)B(t)$, by issuance of public debt. Then, the flow budget constraint of the government is following:

$$B(t+1) = R(t)B(t) + D(t). \tag{48}$$

The government is assumed to keep primary deficit per GDP constant. That is, for all t,

$$D = \frac{D(t)}{Y(t)}. \tag{49}$$

A.4 Equilibrium

A competitive equilibrium of this economy is defined as a set of sequences of state variables, allocations, and price system which satisfy the following conditions in all period t:

1. Given the price system, the allocations solve the household's utility maximization problem;

2. Given the price system, the allocations and state variables solve the firm's profit maximization problem;

3. They satisfy the flow budget constraints of government;

4. They satisfy the following no arbitrage condition on interest rates,

$$1 + f'(k(t)) - \delta = \alpha K(t)^{\alpha-1} L(t)^{1-\alpha} + 1 - \delta = R(t); \tag{50}$$

5. they clear all markets.

A.5 Balanced Growth Path Steady State

Next, we define a balanced growth path steady state (referred to as steady state). We focus on existence of steady states in the numerical analysis. A set of sequences of state variables, allocations, and price system is a balanced growth path steady state if

1. the set of the sequences of the state variables, of allocations, and of price system is a competitive equilibrium, and

2. the gross interest rate is constant forever, $R(t) = R$ for any t.

From the capital market clearing condition and the government's budget constraints, we have

$$\tilde{Z}(t) = \frac{Y(t+1)}{Y(t)}[\tilde{B}(t+1) + \tilde{K}(t+1)], \tag{51}$$

$$\tilde{B}(t+1)\frac{Y(t+1)}{Y(t)} = R(t)\tilde{B}(t) + D, \tag{52}$$

where $\tilde{X}$ is denoted by the ratio of X to GDP. Eliminating $\tilde{B}(t)$ from (51) and (52), we obtain

$$\tilde{Z}(t) - \frac{Y(t+1)}{Y(t)}\tilde{K}(t+1) = R(t)\left[\frac{Y(t-1)}{Y(t)}\tilde{Z}(t-1) - \tilde{K}(t)\right] + D. \tag{53}$$

Finally, in the steady state, (53) can be rewritten as

$$\tilde{Z}(R) = (1+h)(1+n)\left[\frac{\alpha}{R-(1-\delta)}\right] + (1+h)(1+n)\frac{D}{(1+h)(1+n)-R}. \tag{54}$$

R in the steady state must satisfy equation (54). Thus, we obtain equation (19), replacing the left-hand side and the right-hand side of (54) with $\Phi(R)$ and $\Theta(R;D)$, respectively.